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A CRACK ALONG THE CONTINUATION OF THE FACE OF A WEDGE ENVELOPING A HALF-SPACE

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The framework of the plane theory of elasticity is used to investigate the equilibrium of an infinite wedge enveloping a half-space and containing an internal, semi-infinite crack extending along the continuation of one of its faces. The faces of the wedge and the edges of the crack are stress-free. The stresses tend to zero at infinity, but their principal vector and principal moment are different from zero and specified by the condition. An exact solution of the homogeneous, Wiener — Hopf vector equation of the problem in question is constructed, the stress intensity factors at the crack edge are calculated and the asymptotic expressions for the stresses near the wedge edge given.

1. Formulation of the problem. Let an infinite elastic wedge with the opening angle greater than π, contain a crack at $y = 0, x > 1$ (Fig. 1).

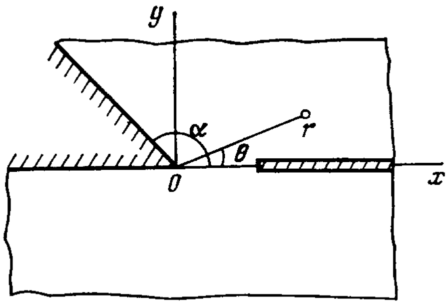

Fig. 1

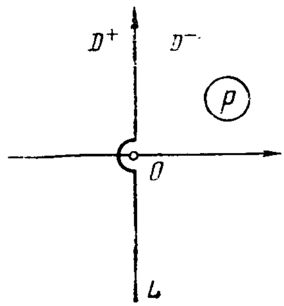

Fig. 2

The wedge faces and crack edges are stress-free. The stresses tend to zero at infinity, but their principal vector and principal moment are different from zero and equal to (X, Y) and M respectively. If follows that the neck $y = 0, 0 < x < 1$ between the wedge $0 < \theta < \alpha$ and the half-space $-\pi < \theta < 0$ transmits the given force and moment

$$\int_0^1 \sigma_\theta(r, 0)\, dr = Y, \quad \int_0^1 \tau_{r\theta}(r, 0)\, dr = X$$

$$\int_0^1 \sigma_\theta(r, 0)\, r dr = M$$

Let us write the boundary conditions and conditions at the singular point of the homogeneous singular problem of the theory of elasticity in question:

$$\begin{aligned}
&\theta = \alpha, \quad \sigma_\theta = \tau_{r\theta} = 0 \quad (0 < \alpha < \pi) \\
&\theta = -\pi, \quad \sigma_\theta = \tau_{r\theta} = 0 \\
&\theta = 0, \quad [\sigma_\theta] = [\tau_{r\theta}] = 0 \\
&\theta = 0, \quad r > 1, \quad \sigma_\theta = \tau_{r\theta} = 0 \\
&\theta = 0, \quad r < 1, \quad [u_\theta] = [u_r] = 0
\end{aligned} \tag{1.1}$$

$$\left[\frac{\partial^2 u_\theta}{\partial r^2}\right]\Bigg|_{\theta=0} \sim \frac{2(1-\nu^2)}{E} \times \tag{1.2}$$

$$\frac{\pi \sin^2 \alpha X - [\alpha(\pi+\alpha) + \sin\alpha(\pi\cos\alpha - \sin\alpha)]Y}{\pi(\alpha^2 - \sin^2\alpha)} \times$$

$$\frac{1}{r^2} + \frac{4(1-\nu^2)}{E} \frac{(\pi+\alpha)\cos\alpha - \sin\alpha}{\pi(\sin\alpha - \alpha\cos\alpha)} \frac{M}{r^3} \quad (r \to \infty)$$

$$\left[\frac{\partial^2 u_r}{\partial r^2}\right]\Bigg|_{\theta=0} \sim \frac{2(1-\nu^2)}{E} \times$$

$$\frac{[\sin\alpha(\pi\cos\alpha + \sin\alpha) - \alpha(\pi+\alpha)]X + \pi\sin^2\alpha Y}{\pi(\alpha^2 - \sin^2\alpha)} \times$$

$$\frac{1}{r^2} - \frac{4(1-\nu^2)}{E} \frac{\sin\alpha}{\sin\alpha - \alpha\cos\alpha} \frac{M}{r^3} \quad (r \to \infty)$$

$$\left[\frac{\partial^2 u_\theta}{\partial r^2}\right]\Bigg|_{\theta=0} \sim -\frac{2(1-\nu^2)}{E} \frac{K_{\mathrm{I}}}{\sqrt{2\pi}(r-1)^{3/2}} \quad (r \to 1+0) \tag{1.3}$$

$$\left[\frac{\partial^2 u_r}{\partial r^2}\right]\Bigg|_{\theta=0} \sim -\frac{2(1-\nu^2)}{E} \frac{K_{\mathrm{II}}}{\sqrt{2\pi}(r-1)^{3/2}}$$

$$\sigma_\theta(r,0) \sim \frac{K_{\mathrm{I}}}{\sqrt{2\pi(1-r)}}, \quad \tau_{r\theta}(r,0) \sim \frac{K_{\mathrm{II}}}{\sqrt{2\pi(1-r)}} \quad (r \to 1-0)$$

$$\sigma_\theta(r,0) \sim \frac{A_1(\alpha)}{r^{1-\lambda_1}}, \quad \tau_{r\theta}(r,0) \sim \frac{A_2(\alpha)}{r^{1-\lambda_1}} \quad (r \to 0, 0 < \alpha \leqslant 2\alpha_* - \pi)$$

$$\sigma_\theta(r,0) \sim \frac{A_1(\alpha)}{r^{1-\lambda_1}} + \frac{B_1(\alpha)}{r^{1-\lambda_2}} \tag{1.4}$$

$$\tau_{r\theta}(r,0) \sim \frac{A_2(\alpha)}{r^{1-\lambda_1}} + \frac{B_2(\alpha)}{r^{1-\lambda_2}} \quad (r \to 0, 2\alpha_* - \pi < \alpha < \pi)$$

Here σ_θ, $\tau_{r\theta}$, σ_r denote the stresses; u_θ and u_r are the displacements; $[N]$ denotes the jump in the value of N; E is the Young's modulus; ν is the Poisson's ratio; K_{I} and K_{II} are the stress intensity factors at the crack edge; $\lambda_j(\alpha)$ $(\beta_j < \alpha < \pi;\ j = 1, 2)$ denotes the single-valued root of the equation $\sin p(\pi+\alpha) - (-1)^j p \sin(\pi+\alpha) = 0$ (p is a complex number) in the strip $0 < \operatorname{Re} p < 1$ $(1/2 < \lambda_j < 1)$; $\beta_1 = 0$, $\beta_2 = 2\alpha_* - \pi$; α_* is the single-valued root of the equation

$$2\varkappa \cos 2\varkappa - \sin 2\varkappa = 0 \quad (\pi/2 < \varkappa < \pi)$$

$A_j(\alpha)$ and $B_j(\alpha)$ are unknown quantitites and $2\alpha_* \approx 257°$.

Conditions (1.2) – (1.4) were formulated using the general assumptions concerning the singular problems of the theory of elasticity (see [1], pp. 51-63).

2. **Solution of the Wiener–Hopf vector equation.** Applying the Mellin transformation to the equations of equilibrium, the conditions of compatibility of deformations, the Hooke's Law relations and to the conditions (1.1), we arrive at the following homogeneous, two-dimensional functional Wiener – Hopf

vector equation:

$$\varphi^+(p) = (p+1)\operatorname{ctg} p\pi\, G(p)\,\varphi^-(p)\ (\lambda_1 - \operatorname{Re} p < 0) \tag{2.1}$$

$$G(p) = \begin{Vmatrix} b(p) + c(p)\,l & c(p)\,m(p) \\ c(p)\,n(p) & b(p) - c(p)\,l \end{Vmatrix}$$

$$b(p) = -\frac{1}{2}\left[\frac{\sin p\pi \sin p\alpha \cos p\alpha}{\cos p\pi\,(p^2\sin^2\alpha - \sin^2 p\alpha)} - 1\right]$$

$$c(p) = -\frac{p\sin p\pi \sin\alpha}{2\cos p\pi\,(p^2\sin^2\alpha - \sin^2 p\alpha)}$$

$$m(p) = (p-1)\sin\alpha,\quad n(p) = -(p+1)\sin\alpha,\quad l = \cos\alpha$$

$$f(p) = l^2 + mn = 1 - p^2\sin^2\alpha$$

$$\varphi^+(p) = (\Phi_1^+(p),\ \Phi_2^+(p)),\quad \varphi^-(p) = (\Phi_1^-(p),\ \Phi_2^-(p))$$

$$\Phi_1^+(p) = \frac{E}{4(1-\nu^2)}\int_1^\infty \left[\frac{\partial^2 u_\theta}{\partial r^2}\right]\Bigg|_{\theta=0} r^{p+1}dr,\quad \Phi_2^+(p) = \frac{E}{4(1-\nu^2)}\times$$

$$\int_1^\infty \left[\frac{\partial^2 u_r}{\partial r^2}\right]\Bigg|_{\theta=0} r^{p+1}dr,\quad \Phi_1^-(p) = \int_0^1 \sigma_\theta(r,0)\,r^p dr$$

$$\Phi_2^-(p) = \int_0^1 \tau_{r\theta}(r,0)\,r^p dr$$

The vector function $\varphi^-(p)$ is analytic in the half-space $\operatorname{Re} p > -\lambda_1$ and $\varphi^+(p)$ is analytic in the half-space $\operatorname{Re} p < 0$, and

$$\Phi_2^+(-1) = 0 \tag{2.2}$$

Let us consider, in the complex p-plane, the contour L consisting of the imaginary axis except a small segment symmetrical about the coordinate origin, and the left half of a circle of small radius with the center at the coordinate origin(Fig. 2). We denote the regions to the left and right of the contour by D^+ and D^- respectively.

It can be shown that the matrix $G(p)$ satisfies the conditions (formulated in [2]) sufficient for representing this matrix on the contour L in the form

$$G(p) = X^+(p)[X^-(p)]^{-1}\quad (p \in L) \tag{2.3}$$

$$X(p) = \begin{cases} X^+(p), & p \in D^+ \\ X^-(p), & p \in D^- \end{cases}$$

$$X(p) = F(p)[I\operatorname{ch}(\sqrt{f}\beta) + U(p)\operatorname{sh}(\sqrt{f}\beta)]$$

$$F(p) = \exp\left[\frac{1}{4\pi i}\int_L \frac{\ln\Delta(t)}{t-p}\,dt\right],\quad \beta(p) = \frac{1}{2\pi i}\int_L \frac{\varepsilon(t)\,dt}{\sqrt{f(t)}\,(t-p)}$$

$$\Delta(p) = \lambda_1(p)\lambda_2(p),\quad \varepsilon(p) = {}^1\!/_2 \ln[\lambda_1(p)/\lambda_2(p)]$$

$$U(p) = [f(p)]^{-1/2}\begin{Vmatrix} l & m(p) \\ n(p) & -l \end{Vmatrix}$$

where $(\lambda_1(p),\ \lambda_2(p)$ are the eigenvalues of the matrix $G(p)$ and I is a unit matrix. From (2.3) we find

$$X(p) \to Q, \quad p \to \infty \tag{2.4}$$

$$Q = \begin{Vmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{Vmatrix}, \quad q = \frac{\sin\alpha}{\pi} \int_0^\infty \frac{\varepsilon(it)\,dt}{\sqrt{1+t^2\sin^2\alpha}}$$

$$[X^-(0)]^{-1} =$$

$$(2aA)^{-1} \begin{Vmatrix} a^2+1-(a^2-1)\cos\alpha & (a^2-1)\sin\alpha \\ (a^2-1)\sin\alpha & a^2+1+(a^2-1)\cos\alpha \end{Vmatrix}$$

$$a = \left[\frac{(\alpha-\sin\alpha)(\pi+\alpha+\sin\alpha)}{(\alpha+\sin\alpha)(\pi+\alpha-\sin\alpha)}\right]^{1/4}, \quad A = \left[\frac{4(\alpha^2-\sin^2\alpha)}{(\pi+\alpha)^2-\sin^2\alpha}\right]^{1/4}$$

$$[X^-(1)]^{-1} = F^{-1} \begin{Vmatrix} \operatorname{ch} s - \operatorname{sh} s & 0 \\ 2\operatorname{tg}\alpha \operatorname{sh} s & \operatorname{ch} s + \operatorname{sh} s \end{Vmatrix}$$

$$s = -\frac{\cos\alpha}{\pi} \int_0^\infty \frac{\varepsilon(it)\,dt}{(t^2+1)\sqrt{1+t^2\sin^2\alpha}},$$

$$F = \exp\left[-\frac{1}{2\pi}\int_0^\infty \frac{\ln\Delta(it)\,dt}{t^2+1}\right]$$

Taking into account the factorization of (2.3) and the known representation [3]

$$p \operatorname{ctg} p\pi = K^+(p)K^-(p), \quad K^\pm(p) = \frac{\Gamma(1 \mp p)}{\Gamma(1/2 \mp p)}$$

where $\Gamma(z)$ is the Euler gamma function, we can rewrite (2.1) thus:

$$p\,[K^+(p)]^{-1}[X^+(p)]^{-1}\varphi^+(p) = (p+1)K^-(p)[X^-(p)]^{-1}\varphi^-(p) \tag{2.5}$$
$$(p \in L)$$

The vector function appearing in the left hand side of (2.5) is analytic in D^+ and the vector function in the right hand side is analytic in D^-. Therefore they are both equal to a single vector function analytic over the whole p-plane.
Let us find this single analytic function. Using (1.3) and letting $p \to \infty$, we obtain [3]

$$\varphi^+(p) \sim (-p/2)^{1/2}v, \quad \varphi^-(p) \sim (2p)^{-1/2}v, \quad v = \begin{bmatrix} K_\mathrm{I} \\ K_\mathrm{II} \end{bmatrix} \tag{2.6}$$

From (2.6), (2.4) and the properties of the function $K^\pm(p)$ it follows that when $p \to \infty$, the left and right hand sides of (2.5) behave as $2^{-1/2}Q^{-1}vp$. Consequently, the single analytic function is equal to $c_0 + c_1 p$ where c_0 and c_1 are vectors to be determined.

According to the condition of the problem we have

$$\varphi^-(0) = \begin{bmatrix} Y \\ X \end{bmatrix} = \chi, \quad \varphi^-(1) = \begin{bmatrix} M \\ \rho \end{bmatrix} = \mu \quad \left(\rho = \int_0^1 \tau_{r\theta}(r,0)\,r\,dr\right) \tag{2.7}$$

where ρ is an unknown quantity. Using (2.7) and (2.5) we find

$$c_0 = \pi^{-1/2}[X^-(0)]^{-1}\chi, \quad c_1 = 4\pi^{-1/2}[X^-(1)]^{-1}\mu - c_0 \tag{2.8}$$

The solution of the functional equation (2.1) can be written in the form

$$\varphi^-(p) = (p+1)^{-1}[K^+(p)]^{-1}X^-(p)(c_0 + c_1 p) \quad (p \in D^-) \tag{2.9}$$

$$\varphi^+(p) = p^{-1}K^+(p)X^+(p)(c_0 + c_1 p) \quad (p \in D^+) \tag{2.10}$$

The unknown ρ can be determined from the condition (2.2) using of (2.10)

$$\rho = F[4(\operatorname{ch} s + \operatorname{sh} s)]^{-1}\left\{(aA)^{-1}\left[(a^2-1)\sin\alpha Y + 2\left(a^2\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\right)X\right] - 8F^{-1}\operatorname{tg}\alpha\operatorname{sh} s\, M\right\} \tag{2.11}$$

3. Stress intensity coefficients at the crack edge. Asymptotics for the stresses near the wedge edge. From (2.9) follows that

$$\varphi^-(p) \sim Qc_1 p^{-1/2} \quad (p \to \infty) \tag{3.1}$$

From (3.1) and (2.6) we obtain the following formula for the stress intensity coefficients:

$$v = (2/\pi)^{1/2}Q\{4[X^-(1)]^{-1}\mu - [X^-(0)]^{-1}\chi\}$$

where the matrices Q, $[X^-(0)]^{-1}$, $[X^-(1)]^{-1}$ are given by the formulas (2.4) and the quantity ρ by (2.11). Setting $\alpha = \pi$ yields a known result [4,5].

Let us inspect the behavior of the stresses $\sigma_\theta(r, 0)$ and $\tau_{r\theta}(r, 0)$ as $r \to 0$. For $p \in D^+$, $\operatorname{Re} p > -\lambda_1$ the analytic function $\varphi^-(p)$ is given by the formula

$$\varphi^-(p) = (p+1)^{-1}[K^-(p)]^{-1}G^{-1}(p)X^+(p)\ (c_0 + c_1 p)$$

$$G^{-1}(p) = \frac{2\cos p\pi}{\sin^2 p(\pi+\alpha) - p^2\sin^2\alpha}\begin{Vmatrix} -h_1(p) & -i_1(p) \\ i_2(p) & h_2(p)\end{Vmatrix}$$

$$h_{1,2}(p) = p\sin\alpha\,(\sin p\pi\cos\alpha \pm p\sin\alpha\cos p\pi) \mp \sin p\alpha\sin p(\pi+\alpha)$$

$$i_{1,2}(p) = p(p \mp 1)\sin p\pi\sin^2\alpha$$

and applying the inversion formula we obtain

$$\begin{bmatrix}\sigma_\theta(r,0)\\ \tau_{r\theta}(r,0)\end{bmatrix} = (2\pi r i)^{+1}\int_\gamma (p+1)^{-1}[K^-(p)]^{-1}G^{-1}(p) \times X^+(p)(c_0 + c_1 p)r^{-p}dp \tag{3.2}$$

where γ denotes a straight line parallel to the imaginary axis and lying in the strip $-\lambda_1 < \operatorname{Re} p < 0$.

Using the theorem of residues, we obtain the following asymptotics $(r \to 0)$ from (3.2):

$$\begin{bmatrix}\sigma_\theta(r,0)\\ \tau_{r\theta}(r,0)\end{bmatrix} \sim \sum_{j=1}^{2} \frac{2\Gamma(^1/_2-\lambda_j)\cos\lambda_j\pi}{\Gamma(2-\lambda_j)}[\sin\lambda_j(\pi+\alpha) -$$

$$(-1)^j\lambda_j\sin\alpha]^{-1}[(\pi+\alpha)\cos\lambda_j(\pi+\alpha)+(-1)^j\sin\alpha]^{-1}\times$$

$$H_jX^+(-\lambda_j)(c_0-\lambda_jc_1)\,r^{\lambda_j-1}$$

$$H_j = \begin{Vmatrix} h_1(\lambda_j) & -i_2(\lambda_j)\\ i_1(\lambda_j) & -h_2(\lambda_j)\end{Vmatrix} \quad (2\alpha_*-\pi<\alpha<\pi)$$

When $0<\alpha\leqslant 2\alpha_*-\pi$, then the term corresponding to $j=2$ vanishes.

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